Distortion of Subgroups of the Generalized Thompson groups $F(n_1, ..., n_k)$

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Groups of the Form $F(n_1, \ldots, n_k)$ have been considered by Bieri, Strebel, and Stein, but the metric properties of these groups have not yet been considered.

- $F(n_1, ..., n_k)$ is the group of piecewise-linear orientation-preserving homeomorphisms of the closed unit interval with finitely-many breakpoints in $\mathbb{Z}[\frac{1}{n_1 n_2 \cdots n_k}]$ and slopes in the cyclic multiplicative group $\langle n_1, n_2, ..., n_k \rangle$ in each linear piece.

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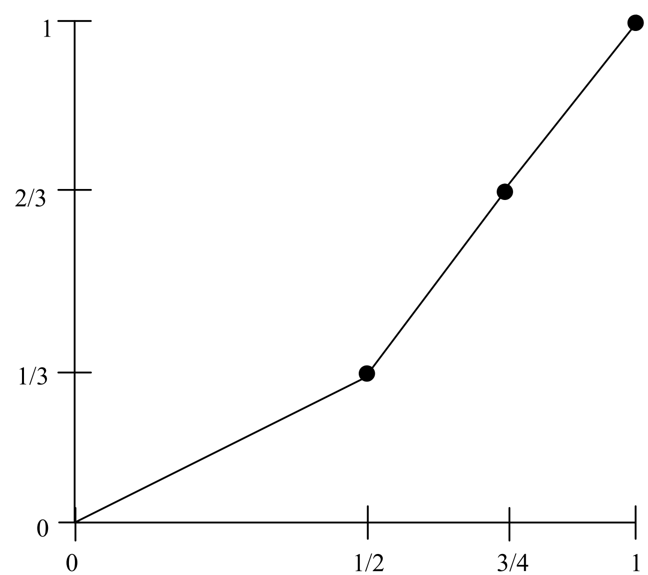
- 

Figure: An element of $F(2, 3)$.

Groups of the Form $F(n_1, ..., n_k)$

We consider $F(n_1, ..., n_k)$ for

- $n_1, ..., n_k \in \{2, 3, 4, ...,\}$,
- $k \in \{2, 3, 4, ...\}$,
- we assume $n_1, ..., n_k$ are relatively prime
- and $n_1 - 1 | n_j - 1$ for all $j \in \{1, ..., k\}$.

Tree-pair diagram representatives

As in the case of F, any element of $F(n_1, ..., n_k)$ can be represented using a $(n_1, ..., n_k)$–ary tree-pair diagram:

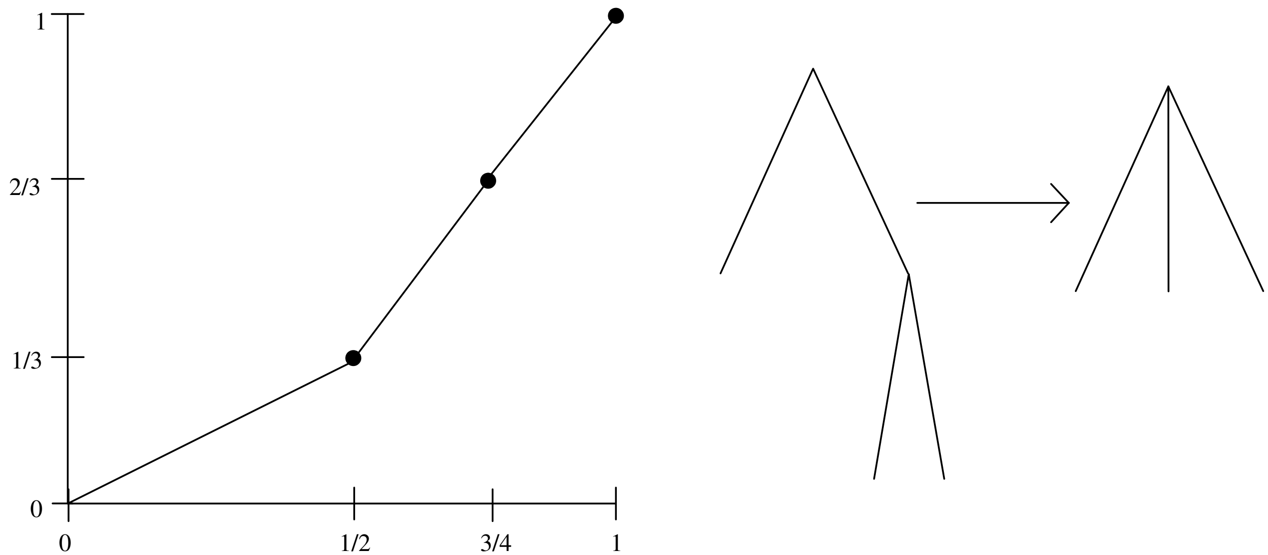


Figure: A $(2, 3)$–ary tree-pair diagram.

Composition

Composition of $(n_1, ..., n_k)$–ary tree-pair diagrams:

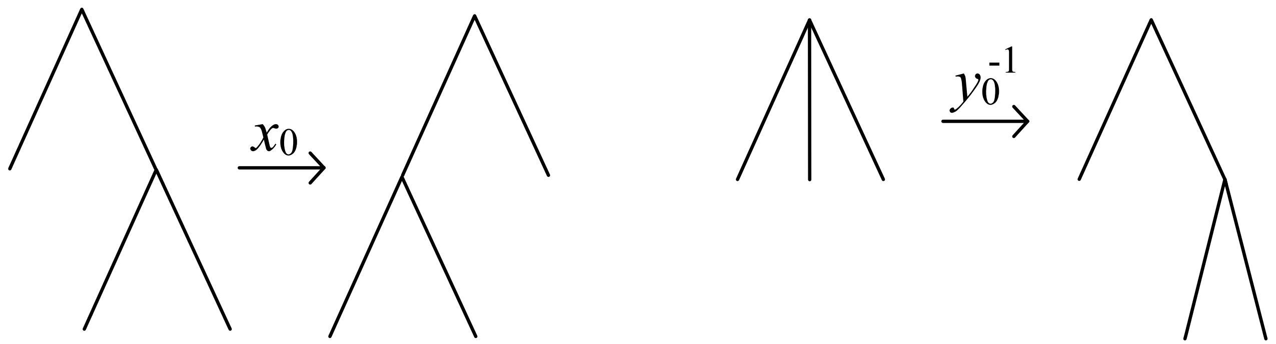


Figure: Composition of two sample elements in $F(2,3)$

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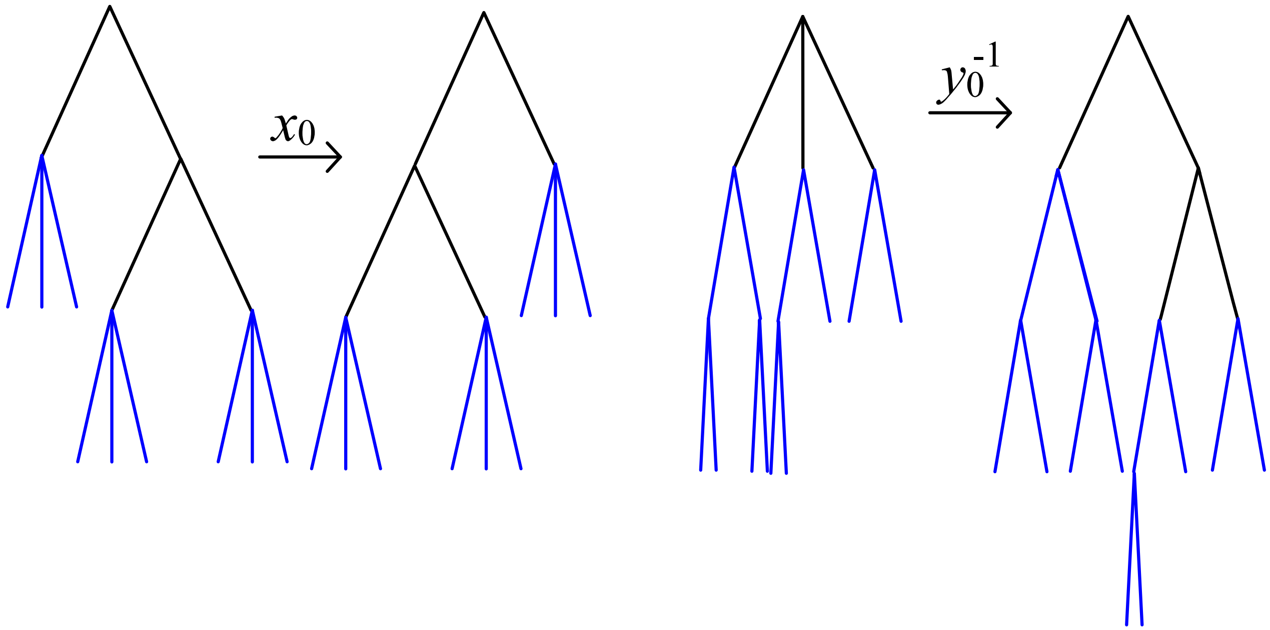

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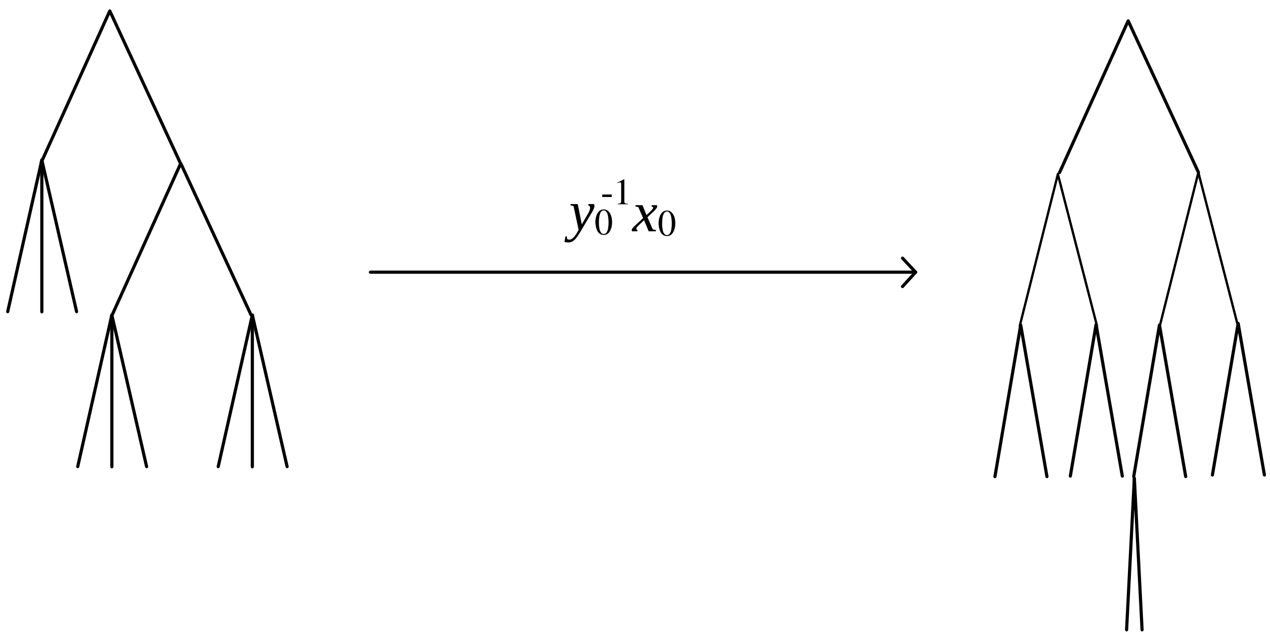

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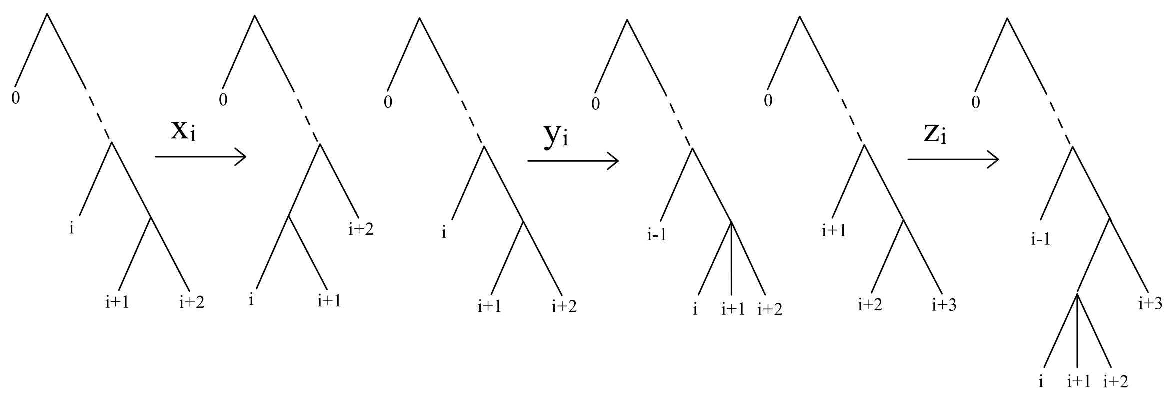
Infinite Presentation (Stein) for Normal Form

The normal form uses the following presentation to allow us to translate directly from tree-pair diagrams to algebraic expressions of elements of $F(n_1, \ldots, n_k)$.

Generators:

$$\{x_0, x_1, \ldots, y_0, y_1, \ldots, z_0, z_1, \ldots\}$$



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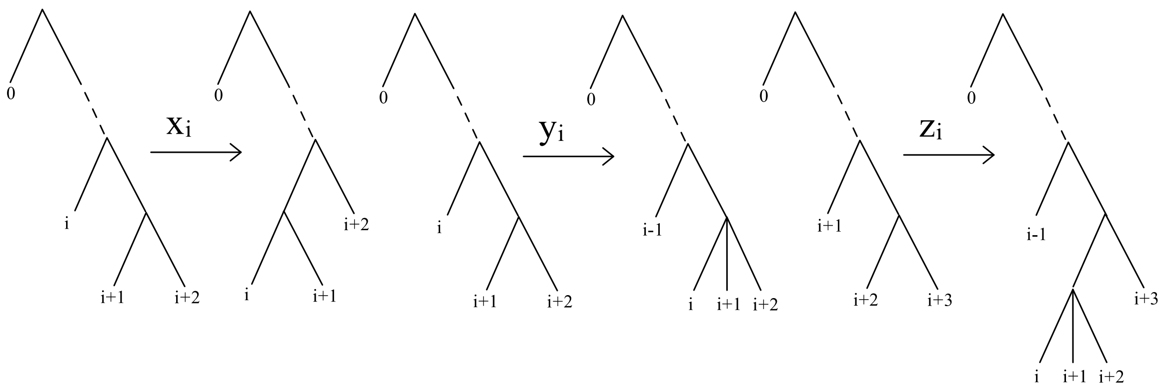

Relators:

1. $x_j x_i = x_i x_{j+1}$
2. $y_j x_i = x_i y_{j+1}$
3. $z_j x_i = x_i z_{j+1}$
4. $x_j z_i = z_i x_{j+2}$
5. $y_j z_i = z_i y_{j+2}$
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for $i < j$ and

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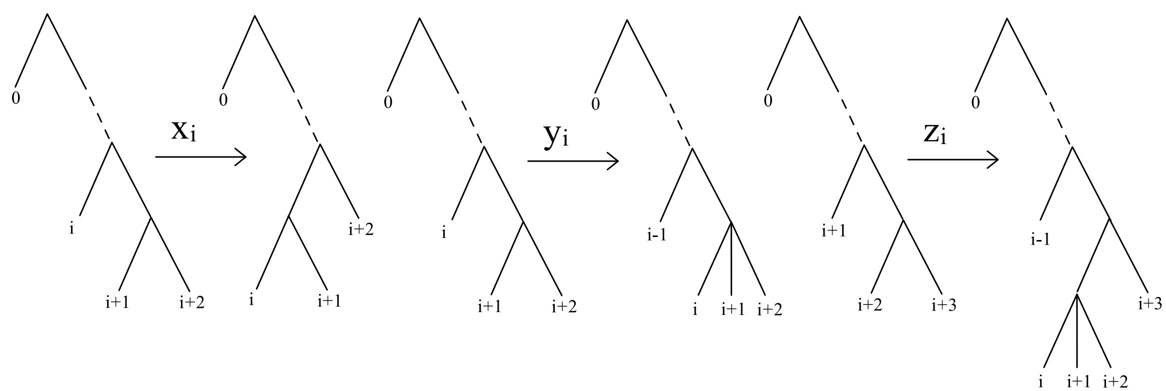

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for $i < j$ and

1. $y_{i+1} z_i = y_i x_{i+1} x_i$
2. $x_i z_{i+1} z_i = z_i x_{i+2} x_{i+1} x_i$

for all i.

Finite Presentation (Stein)

Generators:

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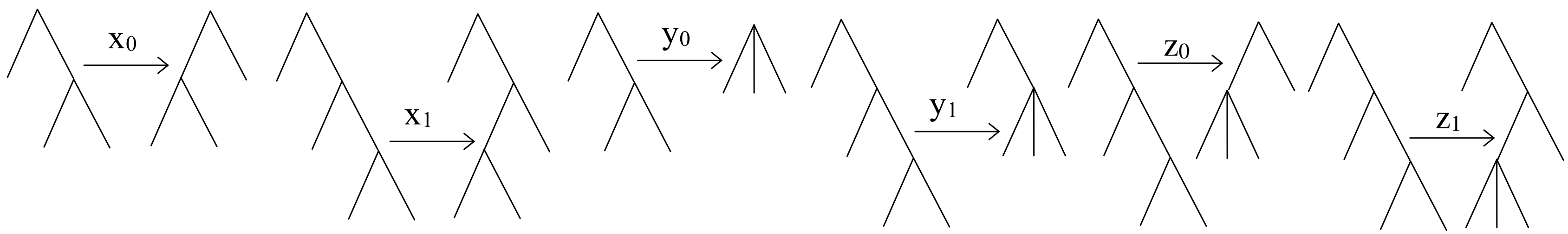

Relators:

1. $x_2x_0 = x_0x_3$
2. $x_3x_1 = x_1x_4$
3. $y_2x_0 = x_0y_3$
4. $y_3x_1 = x_1y_4$
5. $x_1z_0 = z_0x_3$
6. $x_2z_1 = z_1x_4$
7. $y_1z_0 = z_0y_3$
8. $y_2z_1 = z_1y_4$
9. $x_0z_1z_0 = z_0x_2x_1x_0$
10. $x_1z_2z_1 = z_1x_3x_2x_1$

where $x_3 = x_1^{-1}x_2x_1$, $x_4 = x_2^{-1}x_3x_2$, $y_3 = x_1^{-1}y_2x_1$, $y_4 = x_2^{-1}y_3x_2$, and $z_2 = y_3^{-1}y_2x_3x_2$.

The Metric

Using the normal form, we can obtain upper and lower bounds on the metric of $F(n_1, ..., n_k)$ in terms of the number of leaves present in the minimal tree-pair diagram representative.

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Theorem

For a given element $w \in F(n_1, ..., n_k)$, where $L(w)$ denotes the number of leaves in the minimal tree-pair diagram representative of w, there exist fixed c_1, c_2, c_3, c_4 such that

$$c_1 log L(w) + c_2 \leq |w|_{\{x_0, x_1, y_0, y_1\}} \leq c_3 L(w) + c_4$$

Both of these bounds are sharp.

The Metric

An example of an element with length of logarithmic order with respect to the number of leaves in its minimal tree-pair diagram representative:

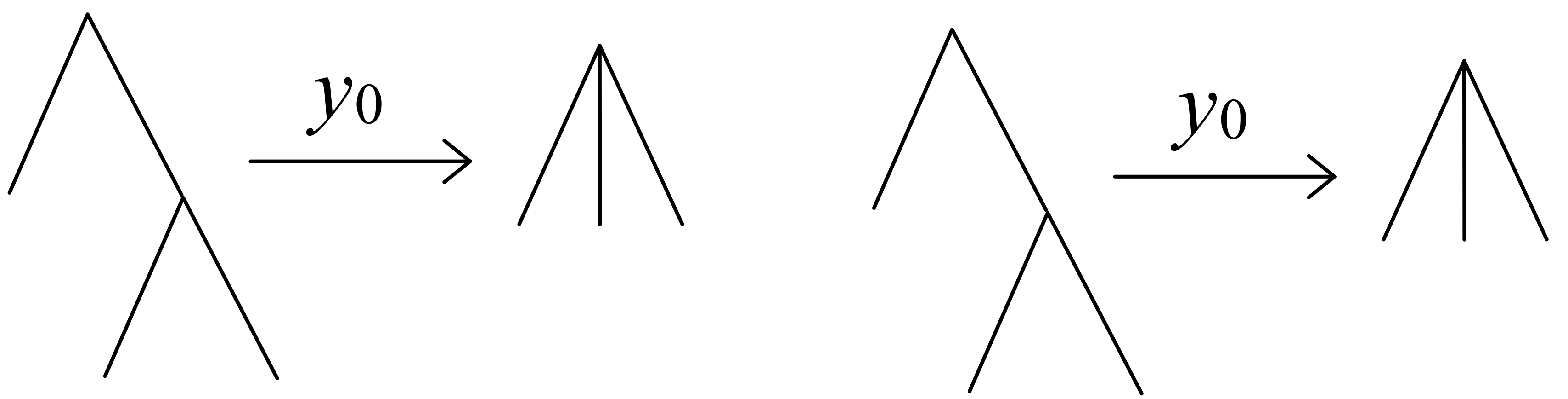

Figure: Computing y_0^n in $F(2,3)$

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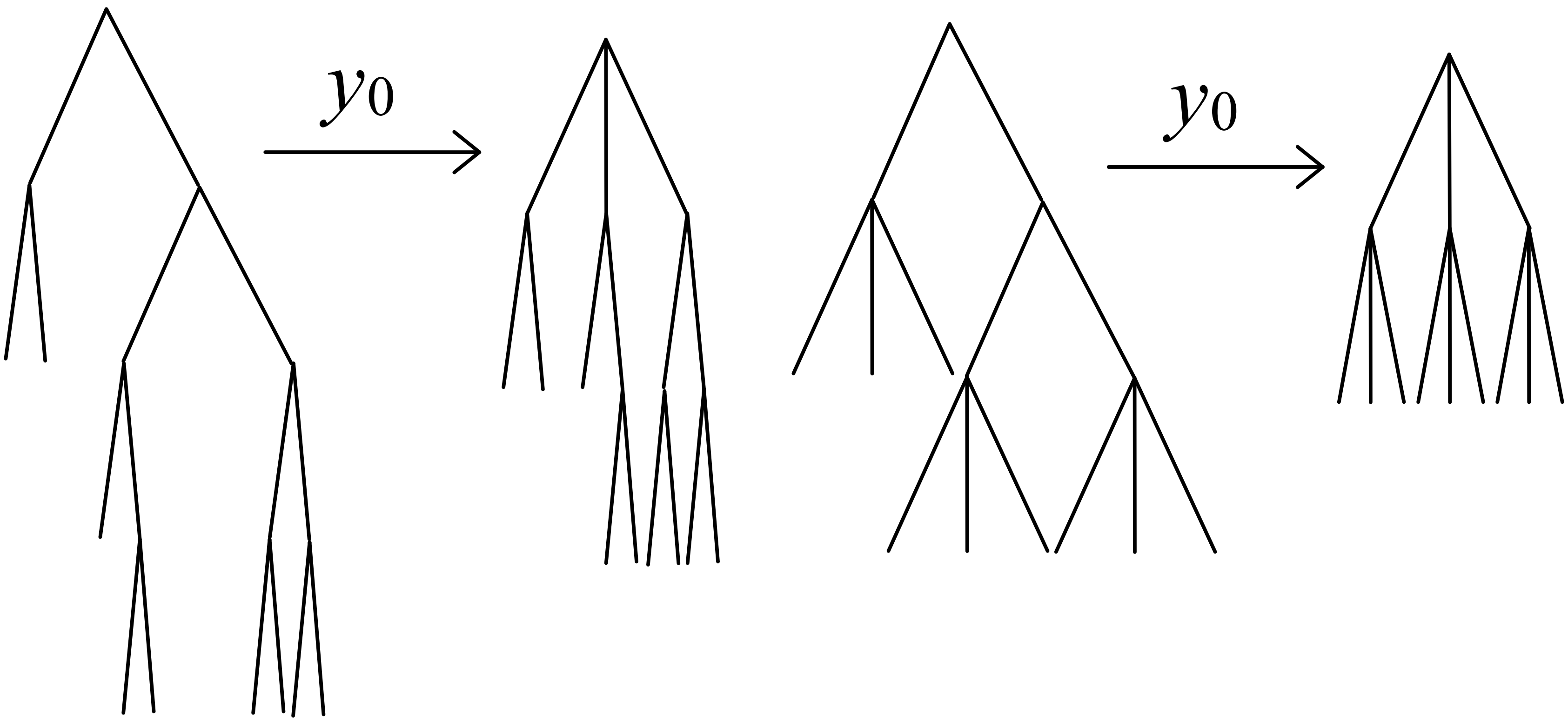

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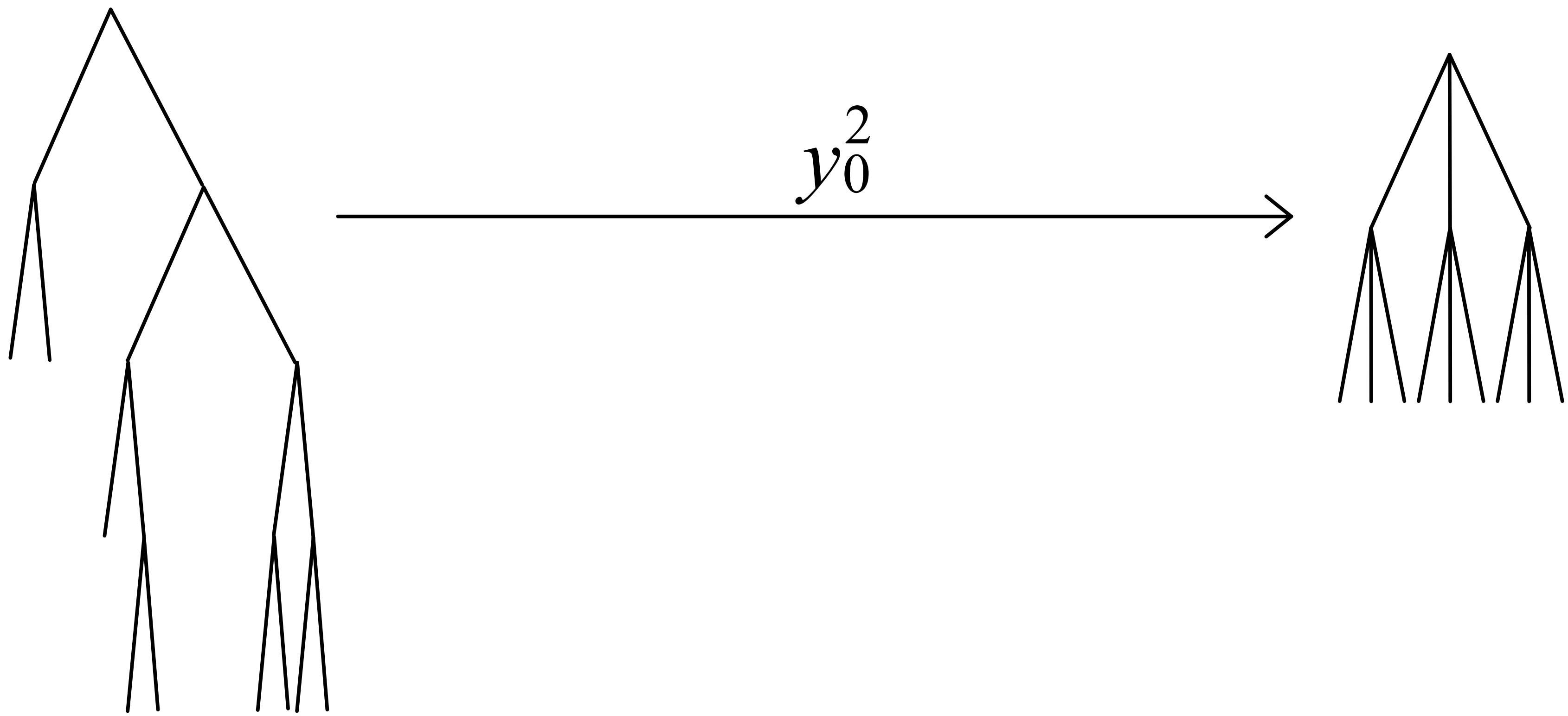

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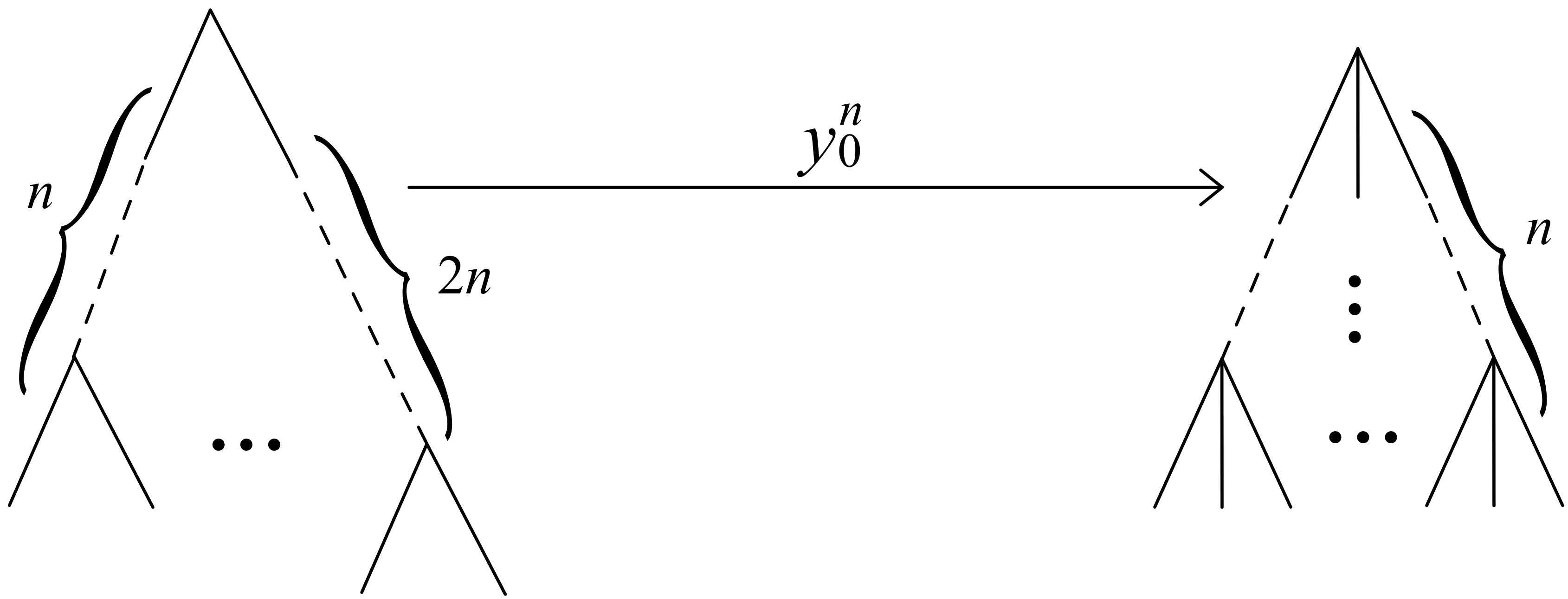
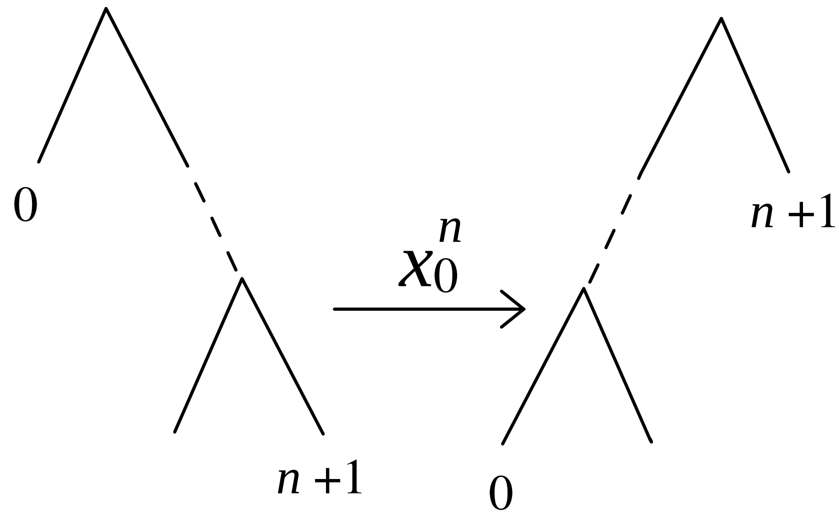


Figure: Computing y_0^n in $F(2,3)$

$L(y_0^n) = 3^n$, but $|y_0^n|_{\{x_0,x_1,y_0,y_1\}} \leq n$.
So the lower bound on our metric estimate is sharp.

The Metric

An example of an element with length of linear order with respect to the number of leaves in its minimal tree-pair diagram representative:



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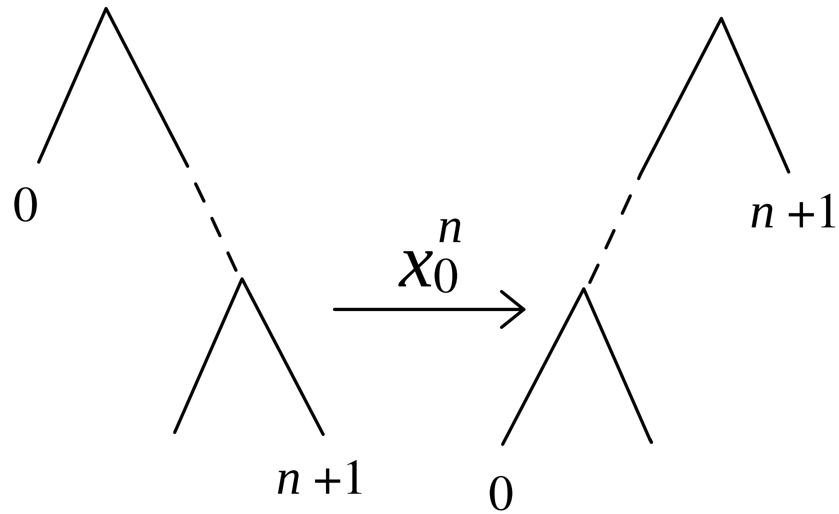

Lemma

If $D(w)$ stands for the depth of w (i.e. the maximum length from the root vertex to any leaf vertex in the minimal tree-pair diagram representative), then $|w|_{\{x_0,x_1,y_0,y_1\}} \geq \frac{D(w)}{3}$.

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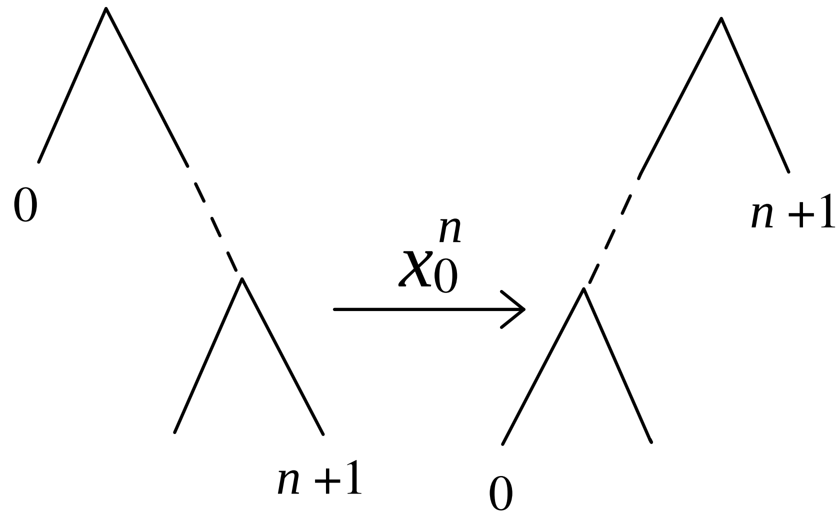

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$L(x_0^n) = D(x_0^n) + 1$, so $\frac{L(x_0^n)-1}{3} \leq |x_0^n|_{\{x_0,x_1,y_0,y_1\}} \leq c_1 L(x_0^n) + c_2$.
So the upper bound on our metric estimate is sharp.

Subgroup Embeddings

Subgroup Embeddings

Theorem

For any n_i *such that there exists* n_j *with* $j \in \{1, ..., k\}$, $i \neq k$*, and* $n_i - 1 | n_j - 1$*,* $F(n_i)$ *is exponentially distorted in* $F(n_1, ..., n_k)$*.*

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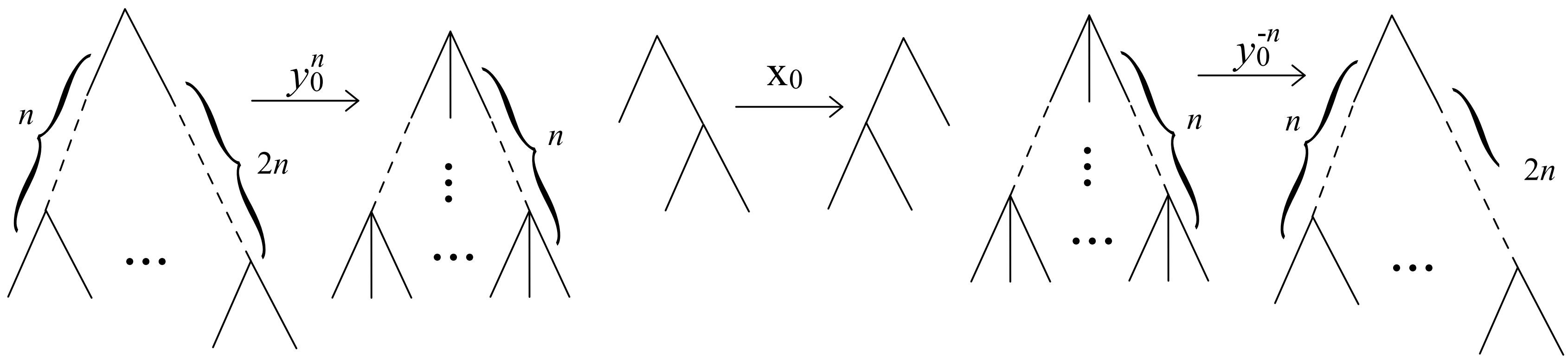

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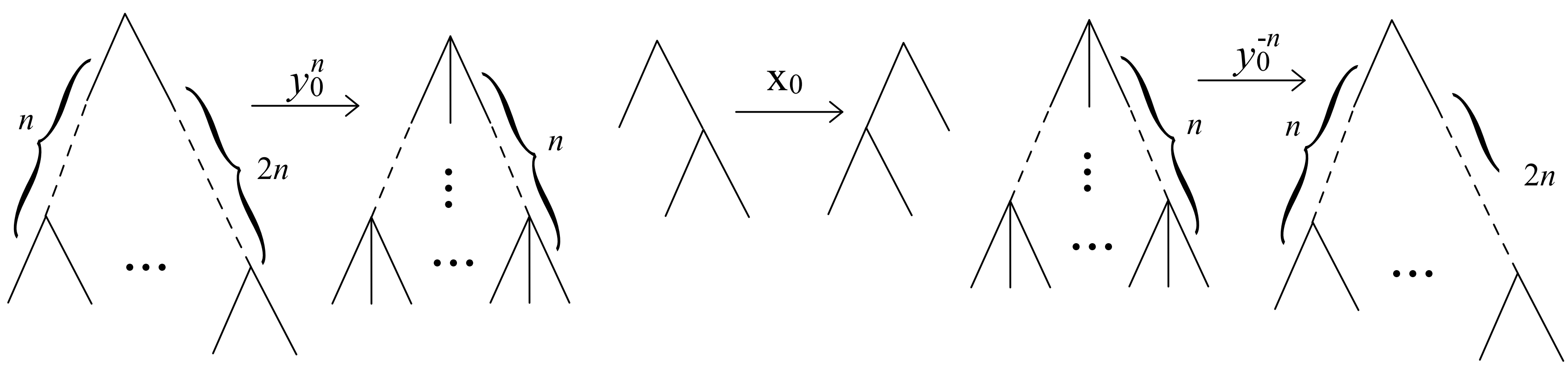

Let $w_n = y_0^{-n} x_0 y_x^n$. $L(w_n)$ is of the order 3^n, so $|w_n|_{\{x_0,x_1\} \in F(2)}$ is of the order 3^n. But $|w_n|_{\{x_0,x_1,y_0,y_1\} \in F(2,3)} \leq 2n + 1$.
So $|w_n|_{\{x_0,x_1\} \in F(2)}$ grows exponentially with respect to $|w_n|_{\{x_0,x_1,y_0,y_1\} \in F(2,3)}$.

Cyclic Subgroups are Quasi-isomorphically Embedded in $F(n_1, \ldots, n_k)$

All cyclic subgroups $\langle x \rangle$ of $F(n_1, \ldots, n_k)$ break down into two cases:

1. $|x^n|_{F(n_1,\ldots,n_k)}$ is quasi-isometric to $L(x^n)$.
2. $|x^n|_{F(n_1,\ldots,n_k)}$ is quasi-isometric to $\log(L(x^n))$.

Cyclic Subgroups are Quasi-isomorphically Embedded in $F(n_1, ..., n_k)$

Definition ((disjoint)leaf sets)

For a given element $x = (T_-, T_+) \in F(n_1, ..., n_k)$*, we let* T_-^* *and* T_+^* *denote the minimal trees that can be obtained from* T_- *and* T_+ *respectively by adding carets until* $T_-^* \equiv T_+^*$*.*
Then the negative leaf set of x *is the set of leaf index numbers*

$$\{i | \text{carets must be added to the leaf } l_i \in T_- \text{ to obtain } T_-*\}$$

We can similarly define the positive leaf set of x*.*

Cyclic Subgroups are Quasi-isomorphically Embedded in $F(n_1, \ldots, n_k)$

Lemma

For $x \in F(n_1, ..., n_k)$, whenever $x = (T_-, T_+)$ has disjoint leaf sets, $D(x^n)$, $L(x^n)$, and $|x^n|_{\langle x \rangle} = n$ are all quasi-isometric.

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Theorem

For $x \in F(n_1, ..., n_k)$ and some fixed non-negative integer N, if $x^n = (T_-, T_+)$ has disjoint leaf sets for all $n \geq N$, then $|x^n|_{\langle x \rangle} = n$ and $|x^n|_{F(n_1, ..., n_k)}$ are quasi-isometric, and both are quasi-isometric to $L(x^n)$.

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Theorem

For $x = (T_-, T_+) \in F(n_1, ..., n_k)$, if the intersection of the leaf sets of x^n is nonempty for all non-negative integers n, then $|x^n|_{\langle x \rangle} = n$ and $|x^n|_{F(n_1, ..., n_k)}$ are quasi-isometric, and both of these values are quasi-isometric to $\log L(x^n)$.

Cyclic Subgroups are Quasi-isomorphically Embedded in $F(n_1, ..., n_k)$

Theorem

For all $x \in F(n_1, ..., n_k)$, where $\langle x \rangle$ represents the cyclic subgroup generated by x, $|x^n|_{F(n_1,...,n_k)}$ is quasi-isometric to $n = |x^n|_{\langle x \rangle}$.